

Protocols and quantum circuits for implementing entanglement concentration in cat state, GHZ-like state and 9 families of 4-qubit entangled states

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Three entanglement concentration protocols (ECPs) are proposed. The first ECP and a modified version of that are shown to be useful for the creation of maximally entangled cat and GHZ-like states from their non-maximally entangled counterparts. The last two ECPs are designed for the creation of maximally entangled $(n+1)$ -qubit state $\frac{1}{\sqrt{2}}(|\Psi_0\rangle|0\rangle + |\Psi_1\rangle|1\rangle)$ from the partially entangled $(n+1)$ -qubit normalized state $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$, where $\langle\Psi_1|\Psi_0\rangle = 0$ and $|\alpha| \neq \frac{1}{\sqrt{2}}$. It is also shown that W, GHZ, GHZ-like, Bell and cat states and specific states from the 9 SLOCC-nonequivalent families of 4-qubit entangled states can be expressed as $\frac{1}{\sqrt{2}}(|\Psi_0\rangle|0\rangle + |\Psi_1\rangle|1\rangle)$ and consequently the last two ECPs proposed here are applicable to all these states. Quantum circuits for implementation of the proposed ECPs are provided and it is shown that the proposed ECPs can be realized using linear optics. Efficiency of the ECPs are studied using a recently introduced quantitative measure (Phys. Rev. A **85**, 012307 (2012)). Limitations of the measure are also reported.

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I. INTRODUCTION

Entanglement plays a crucial role in quantum computation and quantum communication [1–8]. It is essential for realization of various protocols and algorithms [1–8]. Specifically, maximally entangled state is required for realization of nonlocal quantum gates [1], distributed quantum search algorithm [2], teleportation [3], densecoding [4], entanglement based quantum key distribution [5] and entanglement based secure direct quantum communication [6–8]. In these applications, entanglement is generally produced locally and distributed to different parties involved in the communication or computation process. During the transmission, processing and storing a maximally entangled pure state may interact with the environment and become mixed state or less entangled (i.e., non-maximally entangled) pure state. This may happen because of many different reasons. For example, the transmission channel may be noisy. In general, the amount of entanglement is usually reduced during transmission process. Unfortunately, such degradation of entanglement is unavoidable, but for the proper execution of the protocols mentioned above [2–8] we need perfect quantum channels for distribution of ebits. In absence of such a channel it would be sufficient to design a protocol that can convert the non-maximally entangled state back into maximally entangled state. Interestingly, such protocols exist and the protocols are divided into two classes (i) Entanglement concentration protocol (ECP) which can transform a partially entangled pure state (pure non-maximally entangled state) into a maximally entangled state and (ii) Entanglement

purification/distillation protocols (EP) that can transform a mixed non-maximally entangled state into a maximally entangled state. In 1996, Bennet et al. [9] proposed the first ECP. In the same year they proposed an EP, too [10]. In their pioneering work Bennet et al. used collective entanglement concentration procedure (i.e., Schmidt projective method). Since the pioneering works of Bennet et al., several ECPs and EPs are proposed [11–26]. Initially most of the ECPs were proposed for non-maximally entangled Bell state. Specifically, Bose et al. [11], Bandyopadhyay [12], Zhao et al. [13], Yamamoto et al. [14], Sheng et al. [15], Sheng and Zhou [16], Gu et al. [17], Deng [18] proposed ECPs and EPs for partially entangled Bell state. While Bose et al.'s proposal involved entanglement swapping and Bell state measurement and Gu et al.'s proposal involved projective operator valued measurement (POVM), other proposals [13–15] circumvented the use of Bell measurement and POVM and discussed the possibilities of optical implementation of proposed ECPs using polarizing beam splitter (PBS) and wave-plates. In fact, ECPs proposed by Zhao et al. and Yamamoto et al. were experimentally realized in 2003 [27, 28]. It was shown in some of these initial works that the ECPs designed for partially entangled Bell states can be generalized to build ECPs for partially entangled cat (N -partite GHZ) states. Recently, several efforts have been made to extend the applicability of ECPs beyond the production of Bell states. For example, in 2012, Sheng et al. have proposed an ECP for partially entangled arbitrary W state [19]. In 2013, Ling-yan He [20] proposed a single nitrogen vacancy (N-V) center assisted ECP for a specific type of partially entangled W state. Very recently, ECPs for partially entangled GHZ states are proposed by Choudhury and Dhara [21] and Zhou et al. [22], ECPs for 4-qubit cluster state are proposed by Choudhury and Dhara [23], Ting-Ting Xu et al. [24] and Zhou et al. [25], ECP for partially entangled NOON

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states is proposed by Zhou et al. [26]. Clearly, much attention has recently been paid to develop ECPs for partially entangled states other than Bell state. Interestingly, majority of these recent efforts are concentrated toward the construction of ECPs for partially entangled GHZ states [[21, 22] and references therein] and W states [[19, 20] and references therein]. In case of 3-qubit pure states it is well-known that there are only 2 families of entangled states [29] under stochastic local quantum operations assisted by classical communication (SLOCC). These two families of 3-qubit entangled states are referred to as GHZ and W states. Thus the recent trend of developing ECPs for partially entangled GHZ and W states is reasonable. Interestingly, no ECP has yet been proposed to explicitly concentrate GHZ-like states. However, in recent past several applications of maximally entangled GHZ-like states have been reported [30–32]. Keeping this in mind, in the present work we have proposed an ECP for partially entangled cat state and have modified that to develop an ECP for partially entangled GHZ-like state.

Recent success in developing ECPs for both the families of 3-qubit entangled states (i.e., for GHZ and W class) also motivated us to ask: How to concentrate pure states of different families of 4-qubit partially entangled states? Present paper's main objective is to answer this question. The effort is timely as to the best of our knowledge until date for 4-qubit pure states ECPs are proposed only for partially entangled 4-qubit cat state and 4-qubit cluster state. No effort has yet been made to concentrate other families of 4-qubit entangled states. Keeping this in mind, present paper is focused around construction of a general ECP for 9 families of 4-qubit entangled states. Before we discuss further detail of our idea, it would be apt to note that in 2002, Verstraete et al. [33] had shown that 4-qubit pure states can be entangled in 9 different ways under SLOCC. A rigorous proof of this classification of 4-qubit pure states was subsequently provided by Chterental and Djokovic in 2007 [34]. In 2010, a similar SLOCC classification of 4-qubit pure states was obtained using string theory [35]. However, in 2010, another interesting result was reported by Gaur and Wallach [36], in which they had established the existence of uncountable number SLOCC-nonequivalent classes of 4-qubit entangled states. In what follows we will use Verstraete et al.'s [33] classification and propose two ECPs that can concentrate some states from each of the 9 families proposed by Verstraete et al. [33].

Until now different strategies have been used for developing ECPs and EPs. For example, ECPs are proposed using linear optics (specifically, using PBSs and wave plates), cross-Kerr-nonlinearities (i.e., using quantum non-demolition (QND) measurements) [15, 19], entanglement swapping and Bell measurement [11], unitary transformation [13] and quantum electrodynamics (QED) based techniques [37] etc. However, most of the recent works discuss ECP with a perspective where qubits are realized using the polarization of photon. As the qubit can be realized using different systems, such as superconductivity, NMR, photon etc., in what follows we have not restricted ourselves to any specific technology and have presented our protocol in general as a quantum circuit. This makes it applicable to any specific kind of realization

of qubits. Only at the end of the paper we have shown that the present work can be realized using PBSs, wave-plates and photon-detectors. As the proposed protocols do not require anything other than implementation of Bell measurement it can also be realized in other implementations of qubits. For example, it is straight forward to implement the ECPs proposed here using NMR-based approach as Bell measurement is possible in NMR [38].

Rest of the paper is organized as follows. In Section II an ECP for cat state is proposed and it is modified to develop an ECP for GHZ-like state. Quantum circuits for these ECPs are also described. In Section III we propose two ECPs for $(n + 1)$ -qubit normalized states of the form $|\psi\rangle = \alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$, where $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are mutually orthogonal n -qubit states and $|\alpha| \neq \frac{1}{\sqrt{2}}$. In this section we have also shown that each of the 9 families of four qubit states contain some states of the form $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$. In Section IV we have shown that the ECPs proposed here can be realized using linear optics. In Section V efficiencies of the proposed ECPs are discussed using a quantitative measure of ECP introduced by Sheng et al. [15] and finally the paper is concluded in Section VI.

II. ENTANGLEMENT CONCENTRATION PROTOCOLS (ECPs)

In the previous section we have already described the importance of ECPs in quantum information processing. In this section we propose new ECPs for partially entangled cat state and GHZ-like states. To begin with we first propose an ECP for a non-maximally entangled cat state.

A. ECP for partially entangled cat state

A non-maximally entangled Bell-type state may be defined as

$$|\psi\rangle_{\text{Bell}} = (\alpha|00\rangle + \beta|11\rangle)_{12}, \quad (1)$$

where $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha| \neq |\beta|$. Similarly, we may define a non-maximally entangled n -qubit cat state as

$$|\psi\rangle_{\text{cat}} = (\alpha|000 \cdots 0\rangle + \beta|111 \cdots 1\rangle)_{12 \cdots n}. \quad (2)$$

We wish to devise an ECP for $|\psi\rangle_{\text{cat}}$ with the help of non-maximally entangled Bell state $|\psi\rangle_{\text{Bell}}$. For this purpose we introduce the quantum circuit shown in Fig. 1. The circuit is composed of two parts. In the first part (See the left most box of Fig. 1) we produce non-maximally entangled n -qubit cat state $|\psi\rangle_{\text{cat}}$ starting from a non-maximally entangled Bell state $|\psi\rangle_{\text{Bell}}$ and $(n - 2)$ auxiliary qubits each prepared in $|0\rangle$. Working of this part of the circuit may be understood as follows: Assume that we have a non-maximally entangled n -qubit cat state $|\psi\rangle_{\text{cat}}$ and we add an auxiliary qubit (prepared

in $|0\rangle$) with that as $(n+1)^{th}$ qubit and apply a CNOT operation with any one of the qubits of the n -qubit $|\psi\rangle_{\text{cat}}$ state (in Fig. 1 it is the second qubit of n -qubit $|\psi\rangle_{\text{cat}}$) as the control

qubit and the auxiliary qubit as the target qubit. This would yield an $(n+1)$ -qubit $|\psi\rangle_{\text{cat}}$ state as

$$\begin{aligned} \text{CNOT}_{2 \rightarrow n+1}(|\psi\rangle_{\text{cat}} \otimes |0\rangle) &= \text{CNOT}_{2 \rightarrow n+1}((\alpha|000 \dots 0\rangle + \beta|111 \dots 1\rangle)_{12 \dots n} \otimes |0\rangle_{n+1}) \\ &= \text{CNOT}_{2 \rightarrow n+1}(\alpha|000 \dots 00\rangle + \beta|111 \dots 10\rangle)_{12 \dots n+1} \\ &= (\alpha|000 \dots 00\rangle + \beta|111 \dots 11\rangle)_{12 \dots n+1}. \end{aligned} \quad (3)$$

This part of the circuit is not the main component of the proposed ECP and it can be ignored. However, this may be relevant in the following scenario: Assume that we have a machine for generation of maximally entangled Bell state $|\psi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. The machine is not working properly and producing $|\psi\rangle_{\text{Bell}}$ (with a fix but unknown value of α) instead of $|\psi^+\rangle$. In such a scenario we may first use an output of that imperfect Bell state generator and $(n-2)$ auxiliary qubits prepared in $|0\rangle$ to produce $|\psi\rangle_{\text{cat}}$. Once we obtain $|\psi\rangle_{\text{cat}}$ (ei-

ther prepared from imperfect Bell state generator or supplied) we may use an entanglement swapping operation between another output of that machine ($|\psi\rangle_{\text{Bell}}$) and $|\psi\rangle_{\text{cat}}$ to obtain the desired ECP for $|\psi\rangle_{\text{cat}}$ as shown in the right most box of the circuit shown in Fig. 1. This part of the circuit works as follows: As the input of the main part of the circuit for implementation of the ECP is $|\psi_1\rangle = |\psi\rangle_{\text{Bell}} \otimes |\psi\rangle_{\text{cat}}$ using (1) and (3) we can write the input state as

$$\begin{aligned} |\psi_1\rangle &= (\alpha|00\rangle + \beta|11\rangle)_{12} \otimes (\alpha|000 \dots 0\rangle + \beta|111 \dots 1\rangle)_{345 \dots n+2} \\ &= (\alpha^2|00000 \dots 0\rangle + \alpha\beta|00111 \dots 1\rangle + \alpha\beta|11000 \dots 0\rangle + \beta^2|11111 \dots 1\rangle)_{12345 \dots n+2}. \end{aligned} \quad (4)$$

Now after applying the SWAP gate shown in the circuit (i.e., after swapping the second and third qubits of $|\psi\rangle_1$ we obtain

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{\sqrt{2}}[(\alpha^2(|\psi^+\rangle + |\psi^-\rangle))_{13}|000 \dots 0\rangle_{245 \dots n+2} + \alpha\beta(|\phi^+\rangle + |\phi^-\rangle)_{13}|011 \dots 1\rangle_{245 \dots n+2} \\ &\quad + \alpha\beta(|\phi^+\rangle - |\phi^-\rangle)_{13}|100 \dots 0\rangle_{245 \dots n+2} + \beta^2(|\psi^+\rangle - |\psi^-\rangle)_{13}|111 \dots 1\rangle_{245 \dots n+2}] \\ &= |\psi^+\rangle_{13} \frac{(\alpha^2|000 \dots 0\rangle + \beta^2|111 \dots 1\rangle)_{245 \dots n+2}}{\sqrt{2}} + |\psi^-\rangle_{13} \frac{(\alpha^2|000 \dots 0\rangle - \beta^2|111 \dots 1\rangle)_{245 \dots n+2}}{\sqrt{2}} \\ &\quad + |\phi^+\rangle_{13} \frac{\alpha\beta(|011 \dots 1\rangle + |100 \dots 0\rangle)_{245 \dots n+2}}{\sqrt{2}} + |\phi^-\rangle_{13} \frac{\alpha\beta(|011 \dots 1\rangle - |100 \dots 0\rangle)_{245 \dots n+2}}{\sqrt{2}}, \end{aligned} \quad (5)$$

where we have used $|\psi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ and $|\phi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$.

After the application of the SWAP gate a Bell measurement is performed on the first two qubits of $|\psi\rangle_2$. As a consequence of that remaining n -qubits would collapse to one of the four states as shown in Column 2 of Table I. If the outcome of the Bell measurement is $|\psi^\pm\rangle$ then the protocol fails, otherwise depending upon the outcome we apply a single qubit unitary operation as shown in Column 3 of Table I. It is easy to observe that on application of the single qubit unitary operation in both the cases (i.e., if the outcome of Bell measurement is $|\phi^+\rangle$ or $|\phi^-\rangle$) we obtain a maximally entangled n -qubit cat state $\frac{|000 \dots 0\rangle + |111 \dots 1\rangle}{\sqrt{2}}$. Thus the circuit shown in Fig. 1 is equivalent to an ECP for non-maximally entangled cat state. Knowledge of values of α and β are not required in the above scenario where we create the state $|\psi\rangle_{\text{cat}}$ starting from a defective Bell state generator which always produces $|\psi\rangle_{\text{Bell}}$ with same values of α and β . In case of any other scenario, we would require to know the value of α or β and

use that to produce $|\psi\rangle_{\text{Bell}}$. Design of such a circuit is a trivial exercise as $|\psi\rangle_{\text{Bell}}$ may be easily created using a modified EPR circuit where the Hadamard gate is replaced by a single qubit unitary gate which maps $|0\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle$. A specific example of such a circuit is shown in Fig. 2 where the single qubit gate U_1 that maps $|0\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle$ is introduced as

$$U_1 = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \quad (6)$$

In general, the proposed ECP is probabilistic. The existing ECPs contain the same probabilistic nature but the feature is not explicitly mentioned. For example, ECPs proposed in recent works of Choudhury and Dhara [21, 23] are essentially probabilistic. However, they didn't mention the probabilistic nature. To be consistent with the conventional ECPs, we may assume that Alice prepares an n -qubit cat state, keeps the first qubit with herself and sends the remaining $n-1$ qubits to

Outcome of Bell measurement	State of the remaining n qubits	Operation applied on qubit 2	Final state
$ \psi^+\rangle_{13}$	$\frac{(\alpha^2 000\dots 0\rangle + \beta^2 111\dots 1\rangle)_{245\dots n+2}}{\sqrt{2}}$	Protocol fails	
$ \psi^-\rangle_{13}$	$\frac{(\alpha^2 000\dots 0\rangle - \beta^2 111\dots 1\rangle)_{245\dots n+2}}{\sqrt{2}}$	Protocol fails	
$ \phi^+\rangle_{13}$	$\frac{\alpha\beta(011\dots 1\rangle + 100\dots 0\rangle)_{245\dots n+2}}{\sqrt{2}}$	X	$\frac{ 000\dots 0\rangle + 111\dots 1\rangle}{\sqrt{2}}$
$ \phi^-\rangle_{13}$	$\frac{\alpha\beta(011\dots 1\rangle - 100\dots 0\rangle)_{245\dots n+2}}{\sqrt{2}}$	iY	$\frac{ 000\dots 0\rangle + 111\dots 1\rangle}{\sqrt{2}}$

Table I: Relation among Alice's Bell state measurement outcome, cat state and operation applied

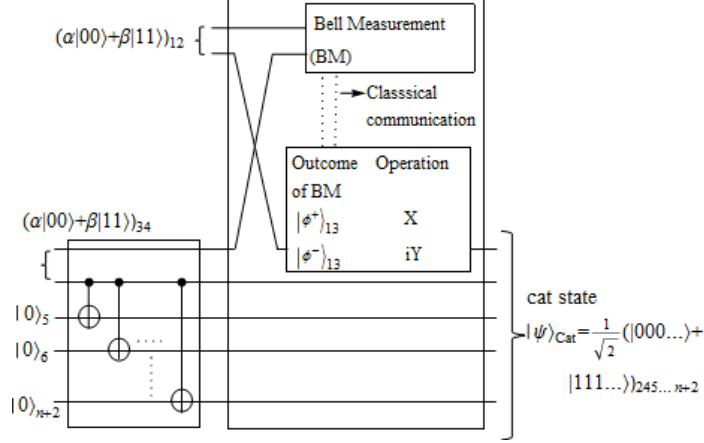


Figure 1: Quantum circuit for ECP of non-maximally entangled cat state $|\psi\rangle_{\text{cat}}$

$n - 1$ parties, say $\text{Bob}_1, \text{Bob}_2, \dots, \text{Bob}_{n-1}$. At a later time the transmitted cat state may be reduced to partially entangled cat state $|\psi\rangle_{\text{cat}}$. In order to concentrate that in the above described protocol Alice prepares $|\psi\rangle_{\text{Bell}}$, swaps her share of the $|\psi\rangle_{\text{cat}}$ with the second qubit of $|\psi\rangle_{\text{Bell}}$, performs a Bell measurement on the first two qubits of her possession and if the protocol succeeds then she applies appropriate unitary

operation (as described in Table I) on the third qubit to obtain a maximally entangled cat state shared between her and $\text{Bob}_1, \text{Bob}_2, \dots, \text{Bob}_{n-1}$. ECPs proposed in the remaining part of this paper can also be illustrated in the similar fashion. However, protocols proposed in the remaining part of the paper are not described in this manner as it is a trivial exercise.

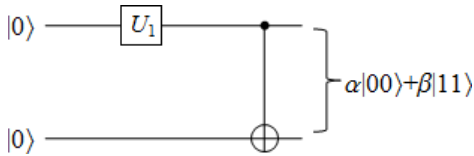


Figure 2: Quantum circuit for generation of $|\psi\rangle_{\text{Bell}} = \alpha|00\rangle + \beta|11\rangle$ where $U_1 = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$

1. Special cases of cat state

Interestingly Bell state and GHZ state are special cases of an n -qubit cat state for $n = 2$ and $n = 3$, respectively. Thus the Bell-measurement based ECP presented here also works as ECP for non-maximally entangled Bell and GHZ states. In 1999, S. Bose et al. [11] proposed a Bell-measurement based

scheme for obtaining the maximally entangled Bell state $|\psi^+\rangle$ from $|\psi\rangle_{\text{Bell}}$. Bose et al.'s scheme can now be viewed as a special case of the above proposed ECP for cat state for $n = 2$. Similarly, for $n = 3$ qubits proposed ECP reduces to an ECP for GHZ state. Very recently a Bell-measurement based ECP for a non-maximally entangled 3-qubit GHZ state is proposed by Choudhury and Dhara [21]. Their ECP can also be viewed as a special case of our ECP for cat state. The scheme proposed here is more general than Choudhury and Dhara [21] scheme for several other reasons, too. For example, α and β are unknown and complex numbers here while in work of Choudhury and Dhara α and β were considered as real and known. Further, complexity of their approach is high as the ECP proposed by Choudhury and Dhara involves many steps that are not essential.

B. ECP for partially entangled GHZ-like state

A maximally entangled GHZ-like state is defined as

$$\frac{|\psi_i 0\rangle \pm |\psi_j 1\rangle}{\sqrt{2}}, \quad (7)$$

where $|\psi_i\rangle, |\psi_j\rangle \in \{|\psi^\pm\rangle, |\phi^\pm\rangle\}$ and $i \neq j$. For example, we may consider a specific GHZ-like state as

$$|\psi\rangle = \frac{|\psi^+ 0\rangle + |\phi^+ 1\rangle}{\sqrt{2}}. \quad (8)$$

Corresponding non-maximally entangled 3-qubit GHZ-like state should be defined as

$$|\psi\rangle_{\text{GHZ-like}} = \alpha|\psi^+ 0\rangle + \beta|\phi^+ 1\rangle. \quad (9)$$

As several applications of maximally entangled GHZ-like states have been reported in recent past [30–32], successful implementation of these applications using GHZ-like state would require an ECP for GHZ-like state. Unfortunately no ECP for GHZ-like state has been proposed until now. Keeping this in mind, we wish to show that a slightly modified version of the ECP described above for cat state works for GHZ-like states. If we start with our defected Bell state generator as before, then with the help of an EPR circuit (a Hadamard gate followed by a CNOT gate) and an auxiliary qubit as shown in the left box of Fig. 3 we can easily produce $|\psi\rangle_{\text{GHZ-like}}$ state. However, such construction of GHZ-like state is not an essential part of the ECP as discussed above. Now after combining the non-maximally entangled GHZ-like state produced as the output of the first block with a non-maximally entangled Bell-type state $|\psi\rangle_{\text{Bell}}$ we obtain

$$\begin{aligned} |\psi_3\rangle &= |\psi\rangle_{\text{Bell}} \otimes |\psi\rangle_{\text{GHZ-like}} = (\alpha|00\rangle + \beta|11\rangle)_{12} \otimes (\alpha|\psi^+ 0\rangle + \beta|\phi^+ 1\rangle)_{345} \\ &= (\alpha^2|00\psi^+ 0\rangle + \alpha\beta|00\phi^+ 1\rangle + \alpha\beta|11\psi^+ 0\rangle + \beta^2|11\phi^+ 1\rangle)_{12345}, \end{aligned} \quad (10)$$

which can be decomposed as

$$\begin{aligned} |\psi_3\rangle &= \frac{1}{\sqrt{2}} [(\alpha^2(|\psi^+\rangle + |\psi^-\rangle)_{15}|0\psi^+\rangle_{234} + \alpha\beta(|\phi^+\rangle + |\phi^-\rangle)_{15}|0\phi^+\rangle_{234} \\ &\quad + \alpha\beta(|\phi^+\rangle - |\phi^-\rangle)_{15}|1\psi^+\rangle_{234} + \beta^2(|\psi^+\rangle - |\psi^-\rangle)_{15}|1\phi^+\rangle_{234}] \\ &= |\psi^+\rangle_{15} \left(\frac{\alpha^2(|000\rangle + |011\rangle) + \beta^2(|101\rangle + |110\rangle)_{234}}{\sqrt{2}} \right) + |\psi^-\rangle_{15} \left(\frac{\alpha^2(|000\rangle + |011\rangle) - \beta^2(|101\rangle + |110\rangle)_{234}}{\sqrt{2}} \right) \\ &\quad + |\phi^+\rangle_{15} \left(\frac{\alpha\beta(|001\rangle + |010\rangle + |100\rangle + |111\rangle)_{234}}{\sqrt{2}} \right) + |\phi^-\rangle_{15} \left(\frac{\alpha\beta(|001\rangle + |010\rangle - |100\rangle - |111\rangle)_{234}}{\sqrt{2}} \right) \\ &= |\psi^+\rangle_{15} \left(\frac{\alpha^2(|000\rangle + |011\rangle) + \beta^2(|101\rangle + |110\rangle)_{234}}{\sqrt{2}} \right) + |\psi^-\rangle_{15} \left(\frac{\alpha^2(|000\rangle + |011\rangle) - \beta^2(|101\rangle + |110\rangle)_{234}}{\sqrt{2}} \right) \\ &\quad + |\phi^+\rangle_{15} \left(\frac{\alpha\beta(|\psi^+ 1\rangle + |\phi^+ 0\rangle)_{234}}{\sqrt{2}} \right) + |\phi^-\rangle_{15} \left(\frac{\alpha\beta(|\psi^- 1\rangle + |\phi^- 0\rangle)_{234}}{\sqrt{2}} \right). \end{aligned} \quad (11)$$

Thus after re-ordering the qubit sequence as 12345 \rightarrow 15234¹ and a Bell measurement on first two qubits of $|\psi_3\rangle$ (as shown in Fig. 3) the state of the remaining 3 qubits would collapse to one of the four states as shown in Column 2 of Table II. If the outcome of Bell measurement is $|\psi^\pm\rangle$ then the protocol fails, otherwise depending upon the outcome we apply a unitary operation as shown in Column 3 of Table II. It is easy to observe that on application of the unitary operation in both the cases (i.e., if the outcome of Bell measurement is

$|\phi^+\rangle$ or $|\phi^-\rangle$) we obtain a maximally entangled 3-qubit GHZ-like state. Thus we have an ECP for GHZ-like state and as far as the main ECP part is concerned this ECP is similar to the ECP designed for cat states with only difference in the choice of qubits to be swapped and to be modified through unitary operation. Thus a quantum circuit designed for ECP of cat states as shown above will also work for GHZ-like states if suitably modified.

III. ECPS FOR QUANTUM STATES OF THE FORM

$$\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$$

In this section we will propose two ECPs for non-maximally entangled $(n+1)$ -qubit state of the following form

$$|\psi\rangle = \alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle, \quad (12)$$

$|\Psi_0\rangle$ and $|\Psi_1\rangle$ are arbitrary n -qubit states that are mutually orthogonal. Before we propose the ECP it would be apt to mention a few words about the relevance of the states of the form $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$. This would justify why are we interested in constructing ECPs for states of this particular form. Clearly GHZ-like, GHZ, Bell and cat states described above are of this form. Further, recently we have shown that

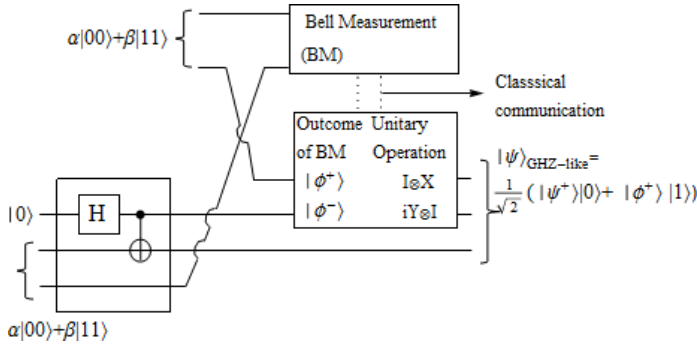


Figure 3: Quantum circuit for ECP of non-maximally entangled GHZ-like state $|\psi\rangle_{\text{GHZ-like}}$.

Outcome of Bell measurement on qubits 1 and 5	Qubits 2, 3 and 4 collapses to	Operation applied on qubits 2 and 3	Final state of qubits 2, 3 and 4
$ \psi^+\rangle_{15}$	$\frac{\alpha^2(000\rangle + 011\rangle)_{234} + \beta^2(101\rangle + 110\rangle)_{234}}{\sqrt{2(\alpha^4 + \beta^4)}}$	Protocol fails	
$ \psi^-\rangle_{15}$	$\frac{\alpha^2(000\rangle + 011\rangle)_{234} - \beta^2(101\rangle + 110\rangle)_{234}}{\sqrt{2(\alpha^4 + \beta^4)}}$	Protocol fails	
$ \phi^+\rangle_{15}$	$\frac{\alpha\beta(\psi^+\rangle + \phi^+\rangle)_{234}}{\sqrt{2}}$	$I \otimes X$	$\frac{(\psi^+\rangle + \phi^+\rangle)_{234}}{\sqrt{2}}$
$ \phi^-\rangle_{15}$	$\frac{\alpha\beta(\psi^-\rangle + \phi^-\rangle)_{234}}{\sqrt{2}}$	$iY \otimes I$	$\frac{(\psi^+\rangle + \phi^+\rangle)_{234}}{\sqrt{2}}$

Table II: Relation among outcome of Bell measurement on qubits 1 and 5 and operation to be applied on the qubits 2, 3 to obtain a maximally entangled GHZ-like state from a non-maximally entangled GHZ-like state.

states of this particular form are useful in bidirectional quantum teleportation [39] and hierarchical quantum communication schemes (e.g., hierarchical quantum information splitting (HQIS), probabilistic HQIS and hierarchical quantum secret sharing (HQSS)) [40]. These recently reported applications and the fact that many well-known entangled states are of these form indicate that the states of the form (12) are of particular importance. Its relevance can be further established by showing that 9 different families of SLOCC-nonequivalent 4-qubit entangled states can be expressed in this form.

In Section I, we have already mentioned that there exist 9 families of 4-qubit entangled states. Following Verstraete et al. [33] we may describe them as

$$\begin{aligned}
G_{abcd} &= \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) \\
&\quad + \frac{b-c}{2}(|0110\rangle + |1001\rangle), \\
L_{abc2} &= \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) \\
&\quad + |0110\rangle, \\
L_{a_2b_2} &= a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle, \\
L_{ab3} &= a(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(|0101\rangle + |1010\rangle) + \frac{a-b}{2}(|0110\rangle + |1001\rangle) \\
&\quad + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle), \\
L_{a_4} &= a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) + (i|0001\rangle + |0110\rangle - i|1011\rangle), \\
L_{a_20_{3\oplus\bar{1}}} &= a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle, \\
L_{0_{5\oplus\bar{3}}} &= |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle, \\
L_{0_{7\oplus\bar{1}}} &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle, \\
L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}} &= |0000\rangle + |0111\rangle.
\end{aligned}$$

Clearly, among 9 families listed above first 6 families (i.e., G_{abcd} , L_{abc2} , $L_{a_2b_2}$, L_{ab3} , L_{a_4} , $L_{a_20_{3\oplus\bar{1}}}$) are parameter-dependent and remaining 3 families are parameter independent. Now we may note that for the specific choices of parameters a , b , c and d , the parameter dependent families yield different quantum states of the form (12) and all the parameter independent families are already in form (12) as shown in last three rows of Table III. Specific examples of interesting quantum states of the form (12) obtained from the parameter dependent families are also shown in Table III. Interestingly,

each of the 9 families contains state of the form (12). Since a state of a family can be transformed to any other state of the family by SLOCC, so if we can construct an ECP for the quantum states of the form (12) in general, that would imply that ECPs can be constructed for a large class of entangled states involving 4-qubits. Here we further note that several applications of the quantum states obtained as examples in 5th Column of Table III are known. For example, applications of Bell state and GHZ state are well-known, recently protocol of quantum dialogue using Q_4 and Q_5 is shown by us in Ref.

Family of states	Values of the parameters a, b, c, d	Corresponding normalized states $ \psi\rangle$	State of the form $\frac{1}{\sqrt{2}}(\Psi_0\rangle 0\rangle + \Psi_1\rangle 1\rangle)$ that belong to the family	Name of the state
G_{abcd}	$a = d = \frac{1}{\sqrt{2}}, b = c = 0$ $a = 1, b = c = d = 0$	$\frac{1}{\sqrt{2}}(0000\rangle + 1111\rangle),$ $\frac{1}{2}(0000\rangle + 1111\rangle + 0011\rangle + 1100\rangle)$ $= \frac{1}{2}[(00\rangle + 11\rangle) \otimes (00\rangle + 11\rangle)]$	$\frac{1}{\sqrt{2}}(000\rangle 0\rangle + 111\rangle 1\rangle),$ $\frac{1}{\sqrt{2}}[(0\rangle 0\rangle + 1\rangle 1\rangle)]$	cat state, Bell state
L_{abc2}	$a = b = 1, c = 0$	$\frac{1}{\sqrt{3}}(0000\rangle + 1111\rangle + 0110\rangle)$	$\frac{1}{\sqrt{3}}[(000\rangle + 011\rangle) 0\rangle + 111\rangle 1\rangle]$	-
L_{a2b2}	$a = 1, b = 0$	$\frac{1}{2}(0000\rangle + 1111\rangle + 0110\rangle + 0011\rangle)$	$\frac{1}{2}[(000\rangle + 011\rangle) 0\rangle + (111\rangle + 001\rangle) 1\rangle]$	-
L_{ab3}	$a = b = 0$	$\frac{1}{2}(0001\rangle + 0010\rangle + 0111\rangle + 1011\rangle)$	$\frac{1}{2}[001\rangle 0\rangle + (000\rangle + 011\rangle + 101\rangle) 1\rangle]$	-
L_{a4}	$a = 0$	$\frac{1}{\sqrt{3}}(0001\rangle + 0110\rangle + 1000\rangle)$	$\frac{1}{\sqrt{3}}[(011\rangle + 100\rangle) 0\rangle + 000\rangle 1\rangle]$	-
$L_{a203\oplus\bar{1}}$	$a = 0$	$ 0011\rangle + 0101\rangle + 0110\rangle =$ $ 0\rangle \otimes \frac{1}{\sqrt{3}}(011\rangle + 101\rangle + 110\rangle)$	$\frac{1}{\sqrt{3}}[11\rangle 0\rangle + (01\rangle + 10\rangle) 1\rangle]$	3-qubit W state
$L_{05\oplus\bar{3}}$	parameter independent	$\frac{1}{2}(0000\rangle + 0101\rangle + 1000\rangle + 1110\rangle)$	$\frac{1}{2}[(000\rangle + 100\rangle + 111\rangle) 0\rangle + 010\rangle 1\rangle]$	Q_4 state [41]
$L_{07\oplus\bar{1}}$	parameter independent	$\frac{1}{2}(0000\rangle + 1011\rangle + 1101\rangle + 1110\rangle)$	$\frac{1}{2}[(000\rangle + 111\rangle) 0\rangle + (101\rangle + 110\rangle) 1\rangle]$	Q_5 state [41]
$L_{03\oplus\bar{1}03\oplus\bar{1}}$	parameter independent	$ 0000\rangle + 0111\rangle = 0\rangle \otimes \frac{1}{\sqrt{2}}(000\rangle + 111\rangle)$	$\frac{1}{\sqrt{2}}(00\rangle 0\rangle + 11\rangle 1\rangle)$	GHZ state

Table III: States corresponding to the 9 families of 4-qubit entangled states. In case of L_{a4} after substituting $a = 0$ local unitary operations are applied to obtain the state $\frac{1}{\sqrt{3}}(|0001\rangle + |0110\rangle + |1000\rangle)$ which belongs to L_{a4} and suitable for the present investigation.

[31]. Further, we have recently shown that states of this form are useful for various kind of hierarchical quantum communication [40]. The general nature and applicability of quantum

states of the form (12) motivated us to construct an ECP for $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$ in general. The same is described in the following section.

A. ECP1 for quantum states of the form $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$

The ECP is described here through the quantum circuit shown in the Fig. 4. In this quantum circuit initial $(n + 1)$ -

qubit state $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$ is combined with $|\psi\rangle_{\text{Bell}}$ and we obtain the combined input state as

$$\begin{aligned}
 |\psi_5\rangle &= (\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle)_{1,2,\dots,n+1} \otimes (\alpha|00\rangle + \beta|11\rangle)_{n+2,n+3} \\
 &= (\alpha^2|\Psi_0\rangle|000\rangle + \alpha\beta|\Psi_1\rangle|100\rangle + \alpha\beta|\Psi_0\rangle|011\rangle + \beta^2|\Psi_1\rangle|111\rangle)_{1,2,\dots,n+1,n+2,n+3}.
 \end{aligned}$$

After swapping the $(n + 1)$ -th with $(n + 2)$ -th qubits we obtain

$$\begin{aligned}
 |\psi_6\rangle &= (\alpha^2|\Psi_0\rangle|000\rangle + \alpha\beta|\Psi_1\rangle|010\rangle + \alpha\beta|\Psi_0\rangle|101\rangle + \beta^2|\Psi_1\rangle|111\rangle)_{1,2,\dots,n,n+2,n+1,n+3} \\
 &= \frac{1}{\sqrt{2}}((\alpha^2|\Psi_0\rangle|0\rangle + \beta^2|\Psi_1\rangle|1\rangle)|\psi^+\rangle + (\alpha^2|\Psi_0\rangle|0\rangle - \beta^2|\Psi_1\rangle|1\rangle)|\psi^-\rangle) \\
 &\quad + \alpha\beta(|\Psi_0\rangle|1\rangle + |\Psi_1\rangle|0\rangle)|\phi^+\rangle + \alpha\beta(|\Psi_0\rangle|1\rangle - |\Psi_1\rangle|0\rangle)|\phi^-\rangle)_{1,2,\dots,n,n+2,n+1,n+3}.
 \end{aligned}$$

Now a Bell measurement is performed on the last two qubits of $|\psi_6\rangle$. If the Bell measurement yields $|\psi^\pm\rangle$ then the protocol fails, but if it yields $|\phi^\pm\rangle$ ($|\phi^-\rangle$) then we can obtain the desired state (i.e., $(\frac{|\Psi_0\rangle|0\rangle + |\Psi_1\rangle|1\rangle}{\sqrt{2}})_{1,2,\dots,n,n+2}$) by applying X (iY) on the $(n + 2)$ -th qubit. This provides a simple, but very useful ECP schemes for quantum states of the form (12) in general. Consequently, we obtain ECP for GHZ-like state, GHZ-state, 9 families of SLOCC-nonequivalent 4-qubit

entangled state, cluster state, cat state etc. In this ECP, knowledge of α, β is required for the construction of $|\psi\rangle_{\text{Bell}}$, but α, β can be complex. In what follows we propose another alternative quantum circuit for ECP of $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$.

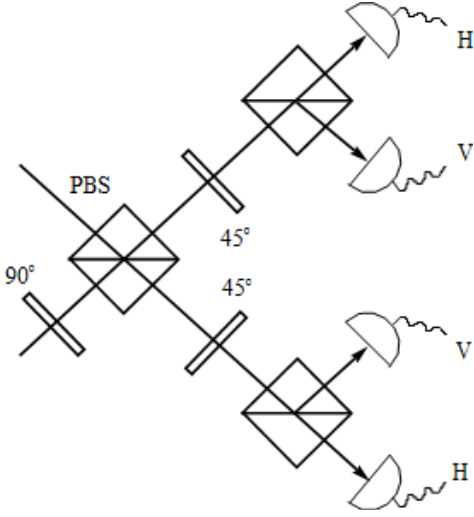


Figure 6: A linear optics based scheme for Bell-state measurement for (a) single photon polarization qubit. The scheme uses polarizing Beam splitters (PBSs) that allows horizontally polarized photon to transmit and reflects the vertically polarized photon, wave plates, and on/off photo detectors [42].

shown in Fig. 6 is elaborately described in Ref. [42]. Here for the completeness of our discussion we may briefly note that when information is encoded using polarization degree of freedom then usually horizontal (H) and vertical (V) polarized states represent $|0\rangle$ and $|1\rangle$, respectively. Thus Bell states can be expressed as $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|HH\rangle \pm |VV\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|HV\rangle \pm |VH\rangle)$. If $|\phi^+\rangle$ enters the optical circuit then the detectors click as either $H_{\text{up}}, V_{\text{down}}$ or $V_{\text{up}}, H_{\text{down}}$ where the subscript up (down) denotes the outcome of top (bottom) two detectors. Similarly, when $|\phi^-\rangle$ enters the optical circuit then the detectors clicks as either $H_{\text{up}}, H_{\text{down}}$ or $V_{\text{up}}, V_{\text{down}}$. Thus $|\phi^+\rangle$ can be distinguished from $|\phi^-\rangle$. However, we cannot distinguish $|\psi^+\rangle$ and $|\psi^-\rangle$ as in both of the cases detectors click as $H_{\text{up}}, V_{\text{up}}$ or $H_{\text{down}}, V_{\text{down}}$. Thus for the Bell-measurement based ECPs proposed here if both of the upper or lower detectors click then the protocol fails, otherwise we apply appropriate unitary operations as described above. Further, the CNOT used in ECP2 can be implemented using optical circuits implemented by J. L. O'Brien et al. [44]. Thus in general ECPs proposed here can be realized optically. However, the applicability of the circuits is not limited to optical realization. For example, these ECPs may be practically realized using NMR as Bell measurement is possible in NMR based technologies [38].

V. EFFICIENCY

Recently Sheng et al. [15] have introduced a quantitative measure of entanglement concentration efficiency of an ECP. They have referred to it as *entanglement transformation efficiency* η and explicitly defined it as

$$\eta = \frac{E_c}{E_0} \quad (13)$$

where, E_0 is the amount of entanglement in the initial partially entangled state and E_c is the amount of entanglement of the state after concentration. Further, they have defined E_c as

$$E_c = P_s \times E_m + (1 - P_s)E' \quad (14)$$

where P_s is the success probability of obtaining the maximally entangled state on execution of the ECP and E_m is the amount of entanglement in the maximally entangled state. Sheng et al. assumed that the measure of entanglement is chosen in such a way that the amount of entanglement in a maximally entangled state is 1, however it may be different in general. For example, we often use a definition of negativity where negativity of maximally entangled Bell state is 0.5 and log negativity is 1. Thus the first (second) term of E_c corresponds to success (failure) of the ECP. Up to this point the definition of η seems clear and straight forward. However, there exists an ambiguity in the definition, it is not clear whether E_0 is the amount of entanglement of the state to be concentrated or that of the entire initial state. To remove this ambiguity we choose E_0 to be the total initial entanglement. This choice naturally implies higher efficiency of single photon assisted ECPs over Bell-type state assisted ECPs (such as the first 2 ECPs of the present paper). Further, a closer look into (13) would reveal that it is neither unique nor easily expendable to the multipartite case. Specifically, the definition does not define which measure of entanglement is to be used for obtaining E_0 . Existence of different measures of bipartite entanglement and the fact that these measures are not monotone of each other makes the definition (13) non-unique. In fact, Sheng et al. [15] have used von Neumann entropy as a measure of entanglement, but von Neumann entropy is a good measure of entanglement for bipartite systems only. This limitation exists for most of the well known measures of entanglement and this fact leads to an interesting question: How to find η for an ECP that is designed for multipartite case. Interestingly, the problem is equivalent to provide a quantitative measure of multipartite entanglement. In last two decades several efforts have been made to introduce measures of multipartite entanglement [45–50]. We may use some of the approaches followed in [45–50] to obtain η for multipartite ECP. To show the dependence of η one choice of entanglement measure we may choose *tangle* [46, 47, 49] as a measure of entanglement. In that case, entanglement of $|\psi\rangle_{\text{Bell}}$ is $4|\alpha\beta|^2$ and success probability for all ECPs presented here and for Sheng et al.'s ECP is $2|\alpha\beta|^2$. Thus η for Sheng protocol and our last protocol (i.e., ECP2) will be $\frac{1}{2}$. Whereas that of our first two ECPs and ECP of Zhao [13] will be $\frac{1}{4}$. As the tangle for a partially entangled GHZ state is same as that of a partially entangled Bell state. Efficiency of our protocol would remain same ($\frac{1}{4}$) for ECP for partially entangled GHZ state. Clearly in all these three cases η is independent of α which is in contrast with the result obtained by Sheng using von Neumann entropy as

a measure of entanglement. Specifically, they had observed η was a function of α . To extend the definition of efficiency η to the multipartite case and to elaborate its dependence on choice of the entanglement measure we may note that in 2004, Yu and Song established [48] that any good measure M_{A-B} of bi-partite entanglement can be generalized to multipartite systems, by considering bipartite partitions of the multipartite system. Yu and Song defined a simple measure of tripartite entanglement as

$$M_{ABC} = \frac{1}{3} (M_{A-BC} + M_{B-AC} + M_{C-AB}), \quad (15)$$

where M_{i-jk} is a measure of entanglement between subsystem i and subsystem jk . M_{i-jk} may be any good measure of bi-partite entanglement (e.g., von Neumann entropy, negativity etc.). Yu and Song's idea was used to measure tripartite entanglement in various systems using different measures of bipartite entanglement e.g., negativity, concurrence, and von Neumann's entropy (cf. [45] and references therein). However, some limitations of the above measure (15) were found and a new measure of tripartite entanglement was introduced by Sabin and Garca-Alcaine by replacing arithmetic mean present in (15) by geometric mean. Thus Sabin and Garca-Alcaine's measure of tripartite entanglement is given as [45]

$$M_{ABC} = (M_{A-BC} M_{B-AC} M_{C-AB})^{\frac{1}{3}}. \quad (16)$$

In what follows we have provided analytic expressions for efficiencies of ECPs proposed here using (13) and (16). To be precise, if we use negativity as a measure of bipartite entanglement then the efficiency of the first two protocols (i.e., protocols assisted by $|\psi\rangle_{\text{Bell}}$) proposed here are as follows

$$\eta_{\text{Bell-type}}^{\text{Bell-type}} = \frac{2|\alpha\beta|^2}{2|\alpha\beta|} = |\alpha\beta|, \quad (17)$$

and

$$\eta_{\text{GHZ-like}}^{\text{Bell-type}} = \frac{2|\alpha\beta|^2}{\sqrt[3]{\frac{1}{4}|\alpha\beta| + |\alpha\beta|}}. \quad (18)$$

Similarly, in case of ECP2 (i.e., for the single qubit assisted protocol) we obtain

$$\eta_{\text{Bell,GHZ}}^{1\text{-qubit}} = 2|\alpha\beta|, \quad (19)$$

and

$$\eta_{\text{GHZ-like}}^{1\text{-qubit}} = \frac{4|\alpha\beta|^2}{\sqrt[3]{2|\alpha\beta|}}. \quad (20)$$

From the above equations it is clear that the ECP2 is more efficient than ECP1. The same is illustrated in Fig. 7. In the left (right) panel of Fig. 7 the variation of efficiency of ECPs proposed for partially entangled Bell and GHZ state (GHZ-like state) with α are shown.

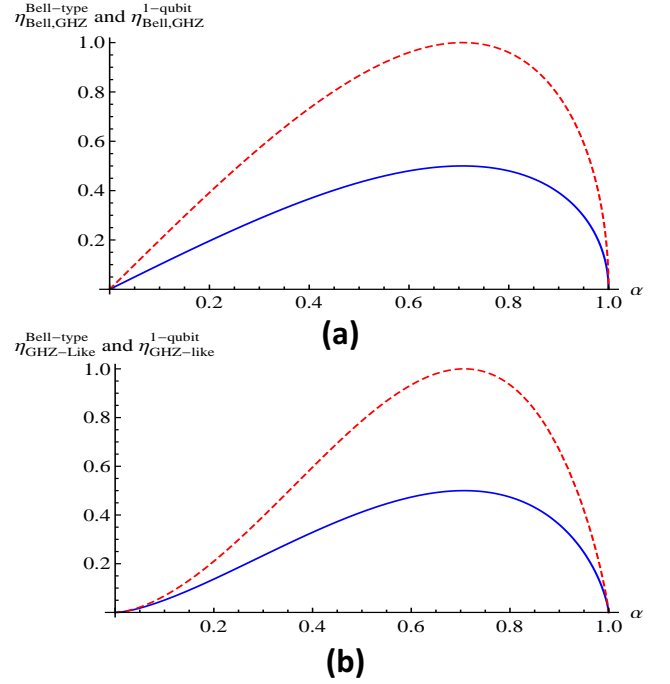


Figure 7: (Color online) Variation of η with α . (a) for Bell and GHZ states and (b) for GHZ-like state. Solid line represents ECPs realized using assistance of $|\psi\rangle_{\text{Bell}}$ and the dashed line represent single-qubit assisted ECP.

VI. CONCLUSION

We have proposed three ECPs in the present paper. The first one is shown to generate a maximally entangled cat state from the corresponding partially entangled state. A modified version of this ECP is also introduced as an ECP for GHZ-like state. ECPs for cat states were proposed earlier, too. However, no ECP for GHZ-like states were proposed until now. Thus this is the first ever ECP reported precisely for GHZ-like state. The last two ECPs are designed for quantum states of the general form $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$. These two protocols are extremely interesting as several applications of the states of the form $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$ are reported in recent past [30–32, 39, 40]. Further, its very important as specific states from the 9 families of SLOCC-nonequivalent 4-qubit entangled states can be described in this form. Thus the proposed ECPs are valid for the 9 families of 4-qubit entangled states. Further, partially entangled cat-like, GHZ-like, GHZ, W and Bell states can also be expressed as $\alpha|\Psi_0\rangle|0\rangle + \beta|\Psi_1\rangle|1\rangle$. The ECPs are not described in the usual style, rather they are described as quantum circuit. From the quantum circuits described above one can clearly see that all the ECPs proposed here require local measurement, classical communication and

post selection. According to Vedral et al. [51] these are the basic steps required by any good ECP or EP. Further, the efficiency of the proposed protocols are discussed in detail using a quantitative measure of efficiency that was recently introduced by Sheng et al. [15]. Apparently any Bell-type state assisted ECP (e.g., the first two ECPs of the present paper and the Zhao et al. proposal [13]) will have lesser efficiency compared to linear optics-based single qubit assisted ECPs [15]. Again if we go beyond linear optics and use nonlinear resources then the efficiency would increase further. However, this parameter cannot be considered as a basis of choosing ECPs as the measure of efficiency introduced by Sheng et al. [15] is really a weak measure. Keeping these in mind and the

fact that the proposed ECPs that are applicable to a large class of quantum states of practical interest are experimentally realizable using linear optical resources and NMR, we conclude the paper with an expectation that experimentalists will find it interesting to implement the ECPs proposed here.

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